

# Spectral properties of empirical covariance matrices for data with power-law tails

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We present an analytic method for calculating spectral densities of empirical covariance matrices for correlated data. In this approach the data is represented as a rectangular random matrix whose columns correspond to sampled states of the system. The method is applicable to a class of random matrices with radial measures including those with heavy (power-law) tails in the probability distribution. As an example we apply it to a multivariate Student distribution.

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## I. INTRODUCTION

Random matrix theory provides a useful tool for description of systems with many degrees of freedom. A large spectrum of problems in physics [1], telecommunication, information theory [2–5] and quantitative finance [6–13] can be naturally formulated in terms of random matrices.

In this paper we apply random matrix theory to calculate the eigenvalue density of the empirical covariance matrix. Statistical properties of this matrix play an important role in many empirical applications. More precisely, the problem which we shall discuss here can be generally formulated in the following way. Consider a statistical system with  $N$  correlated random variables. Imagine that we do not know *a priori* correlations between the variables and that we try to learn about them by sampling the system  $T$  times. Results of the sampling can be stored in a rectangular matrix  $\mathbf{X}$  containing empirical data  $X_{it}$ , where the indices  $i=1, \dots, N$  and  $t=1, \dots, T$  run over the set of random variables and measurements, respectively. If the measurements are uncorrelated in time the two-point correlation function reads

$$\langle X_{i_1 t_1} X_{i_2 t_2} \rangle = C_{i_1 i_2} \delta_{t_1 t_2}, \quad (1)$$

where  $\mathbf{C}$  is called correlation matrix or covariance matrix. For simplicity assume that  $\langle X_{ii} \rangle = 0$ . If one does not know  $\mathbf{C}$  one can try to reconstruct it from the data  $\mathbf{X}$  using the empirical covariance matrix

$$c_{ij} = \frac{1}{T} \sum_{t=1}^T X_{it} X_{jt}, \quad (2)$$

which is a standard estimator of the correlation matrix. One can think of  $\mathbf{X}$  as of an  $N \times T$  random matrix chosen from the matrix ensemble with some prescribed probability measure  $P(\mathbf{X})D\mathbf{X}$ . The empirical covariance matrix:

$$\mathbf{c} = \frac{1}{T} \mathbf{X} \mathbf{X}^T \quad (3)$$

depends thus on  $\mathbf{X}$ . Here  $\mathbf{X}^T$  stands for the transpose of  $\mathbf{X}$ . For the given random matrix  $\mathbf{X}$  the eigenvalue density of the empirical matrix  $\mathbf{c}$  is

$$\rho(\mathbf{X}, \lambda) \equiv \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \lambda_i(\mathbf{c})), \quad (4)$$

where  $\lambda_i(\mathbf{c})$ 's denote eigenvalues of  $\mathbf{c}$ . Averaging over all random matrices  $\mathbf{X}$ ,

$$\rho(\lambda) \equiv \langle \rho(\mathbf{X}, \lambda) \rangle = \int \rho(\mathbf{X}, \lambda) P(\mathbf{X}) D\mathbf{X}, \quad (5)$$

we can find the eigenvalue density of  $\mathbf{c}$  which is representative for the whole ensemble of  $\mathbf{X}$ . We are interested in how the eigenvalue spectrum of  $\mathbf{c}$  is related to that of  $\mathbf{C}$  [14–16]. Clearly, as follows from (1), the quality of the information encoded in the empirical covariance matrix  $\mathbf{c}$  depends on the number of samples or more precisely on the ratio  $r=N/T$ . Only in the limit  $T \rightarrow \infty$ , that is for  $r \rightarrow 0$ , the empirical matrix  $\mathbf{c}$  perfectly reproduces the real covariance matrix  $\mathbf{C}$ . Recently a lot of effort has been made to understand the statistical relation between  $\mathbf{c}$  and  $\mathbf{C}$  for finite  $r$ . This relation plays an important role in the theory of portfolio selection where  $X_{it}$  are identified with normalized stocks' returns and  $\mathbf{C}$  is the covariance matrix for interstock correlations. It is a common practice to reconstruct the covariance matrix from historical data using the estimator (2). Since the estimator is calculated for a finite historical sample it contains a statistical noise. The question is how to optimally clean the spectrum of the empirical matrix  $\mathbf{c}$  from the noise in order to obtain a best quality estimate of the spectrum of the underlying exact covariance matrix  $\mathbf{C}$ . One can consider a more general problem, where in addition to the correlations between the degrees of freedom (stocks) there are also temporal correlations between measurements [17],

$$\langle X_{i_1 t_1} X_{i_2 t_2} \rangle = C_{i_1 i_2} A_{t_1 t_2}, \quad (6)$$

given by an autocorrelation matrix  $\mathbf{A}$ . If  $\mathbf{X}$  is a Gaussian random matrix, or more precisely if the probability measure  $P(\mathbf{X})D\mathbf{X}$  is Gaussian, then the problem is analytically solv-

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able in the limit of large matrices [17–20]. One can derive then an exact relation between the eigenvalue spectrum of the empirical covariance matrix  $\mathbf{c}$  and the spectra of the correlation matrices  $\mathbf{A}$  and  $\mathbf{C}$ .

In this paper we present an analytic solution for a class of probability measures  $P(\mathbf{X})D\mathbf{X}$  for which the marginal distributions of individual degrees of freedom have power-law tails:  $p(X_{it}) \sim X_{it}^{-1-\nu}$  which means that the cumulative distribution function falls like  $X_{it}^{-\nu}$ . Such kind of distributions has been discussed previously [21,22] but, up to our knowledge, the spectral density of  $\mathbf{c}$  remained unattainable analytically. The motivation to study such systems comes from the empirical observation that stocks' returns on financial markets undergo non-Gaussian fluctuations with power-law tails. The observed value of the power-law exponent  $\nu \approx 3$  seems to be universal for a wide class of financial assets [23–25]. Random matrix ensembles with heavy tails have been recently considered for  $0 < \nu < 2$  using the concept of Lévy stable distributions [26–28]. Here we will present a method which extrapolates also to the case  $\nu > 2$ , being of particular interest for financial markets.

We will study here a model which on the one hand preserves the structure of correlations (6) and on the other hand has power-law tails in the marginal probability distributions for individual matrix elements. More generally, we will calculate the eigenvalue density of the empirical covariance matrix  $\mathbf{c}$  (3) for random matrices  $\mathbf{X}$  which have a probability distribution of the form

$$P_f(\mathbf{X})D\mathbf{X} = \mathcal{N}^{-1} f(\text{Tr } \mathbf{X}^T \mathbf{C}^{-1} \mathbf{X} \mathbf{A}^{-1}) D\mathbf{X}, \quad (7)$$

where  $D\mathbf{X} = \prod_{i,t=1}^{N,T} dX_{it}$  is a volume element. The normalization constant  $\mathcal{N}$ ,

$$\mathcal{N} = \pi^{d/2} (\text{Det } \mathbf{C})^{T/2} (\text{Det } \mathbf{A})^{N/2} \quad (8)$$

and the parameter  $d = NT$  have been introduced for convenience. The function  $f$  is an arbitrary non-negative function such that  $P(\mathbf{X})$  is normalized:  $\int P(\mathbf{X})D\mathbf{X} = 1$ .

In particular we will consider an ensemble of random matrices with the probability measure given by a multivariate Student distribution,

$$P_\nu(\mathbf{X})D\mathbf{X} = \frac{\Gamma\left(\frac{\nu+d}{2}\right)}{\mathcal{N}\Gamma\left(\frac{\nu}{2}\right)\sigma^d} \left(1 + \frac{1}{\sigma^2} \text{Tr } \mathbf{X}^T \mathbf{C}^{-1} \mathbf{X} \mathbf{A}^{-1}\right)^{-(\nu+d)/2} D\mathbf{X}. \quad (9)$$

The two-point correlation function can be easily calculated for this measure,

$$\langle X_{i_1 t_1} X_{i_2 t_2} \rangle = \frac{\sigma^2}{\nu-2} C_{i_1 i_2} A_{t_1 t_2}. \quad (10)$$

We see that for  $\sigma^2 = \nu - 2$  and for  $\nu > 2$  the last equation takes the form (6). With this choice of  $\sigma^2$  the two-point function becomes independent on  $\nu$ , however the formula for the probability measure (9) breaks down at  $\nu = 2$  and cannot be extrapolated to the range  $0 < \nu \leq 2$ . An alternative and actually a more conventional choice is  $\sigma^2 \equiv \nu$  which extrapolates

easily to this range. In this case one must remember that for  $\nu > 2$  the exact covariance matrix is given by  $\frac{\nu}{\nu-2} \mathbf{C}$ , where  $\mathbf{C}$  is the matrix in Eq. (9) with  $\sigma^2 = \nu$ . We will stick to this choice in the remaining part of the paper.

The marginal probability distribution for a matrix element  $X_{it}$  can be obtained by integrating out all other degrees of freedom from the probability measure  $P(\mathbf{X})D\mathbf{X}$ . One can see that for the Student probability measure (9) the marginal distributions of individual elements have by construction power-law tails. For example, if  $\mathbf{C}$  is diagonal  $\mathbf{C} = \text{Diag}(C_1^2, \dots, C_N^2)$  and  $\mathbf{A} = \mathbf{1}_T$  then the marginal probability distributions can be found exactly for each element of the matrix  $\mathbf{X}$ ,

$$p_i(X_{it}) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}C_i} \left(1 + \frac{X_{it}^2}{\nu C_i^2}\right)^{-(\nu+1)/2}. \quad (11)$$

The distributions  $p_i$  fall like  $\sim X_{it}^{-1-\nu}$  for large  $X_{it}$  with amplitudes which depend on the index  $i$  and are independent of  $t$ . If one thinks of a stock market, this means that stocks' returns have the same tail exponent but different tail amplitudes. The independence of  $t$  means that the distributions  $p_i(X_{it})$  are stationary. More generally, for any  $\mathbf{C}$  and for  $\mathbf{A}$  which is translationally invariant  $A_{t_1 t_2} = A(|t_1 - t_2|)$  the marginal distributions of entries  $X_{it}$  can be shown to have power-law tails with the same exponent  $\nu$  for all  $X_{it}$  and tail coefficients which depend on  $i$  and are independent of  $t$ , exactly expected from stocks' returns on a financial market.

The main purpose of this paper is to calculate the spectral density of the empirical covariance matrix  $\mathbf{c}$  for the Student distribution (9). The method is similar to the one presented in Refs. [29–33] for a square Hermitian matrix. It consists in an observation that every quantity averaged over the probability distribution having the form (7) can be first averaged over  $(d-1)$  “angular” variables and then of a “radial” variable. This shall be shortly presented in Secs. II and III. In Sec. IV the main equation for the eigenvalue density of  $\mathbf{c}$  for the radial ensemble (7) with an arbitrary radial profile  $f$  shall be given. Section V contains results for the Student distribution (9) including some special cases.

## II. RADIAL MEASURES

The radial measure (7) depends on one scalar function  $f = f(x^2)$  of a real positive argument. In this section we shall develop a formalism to calculate the eigenvalue spectrum  $\rho_f(\lambda)$  of the empirical covariance matrix (3) for such radial ensembles. The calculation can be simplified by noticing that the dependence of  $\rho_f(\lambda)$  on the matrices  $\mathbf{C}$  and  $\mathbf{A}$  actually reduces to a dependence on their spectra. This follows from an observation that for a radial measure (7) the integral (5)

defining the eigenvalue density is invariant under simultaneous transformations,

$$\begin{aligned} \mathbf{C} &\rightarrow \tilde{\mathbf{C}} = \mathbf{O}\mathbf{C}\mathbf{O}^\tau, \\ \mathbf{A} &\rightarrow \tilde{\mathbf{A}} = \mathbf{Q}^\tau\mathbf{A}\mathbf{Q}, \\ \mathbf{X} &\rightarrow \tilde{\mathbf{X}} = \mathbf{O}\mathbf{X}\mathbf{Q}^\tau, \end{aligned} \quad (12)$$

where  $\mathbf{O}$ ,  $\mathbf{Q}$  are orthogonal matrices of size  $N \times N$  and  $T \times T$ , respectively. Choosing the orthogonal transformations  $\mathbf{O}$  and  $\mathbf{Q}$  in such a way that  $\tilde{\mathbf{C}}$  and  $\tilde{\mathbf{A}}$  become diagonal:  $\tilde{\mathbf{C}} = \text{Diag}(C_1^2, \dots, C_N^2)$ ,  $\tilde{\mathbf{A}} = \text{Diag}(A_1^2, \dots, A_T^2)$  with all  $C_i$ 's and  $A_i$ 's being positive, we see that  $\rho_f(\lambda)$  depends on the matrices  $\mathbf{C}$  and  $\mathbf{A}$  indeed only through their eigenvalues. Therefore, for convenience we shall assume that  $\mathbf{C}$  and  $\mathbf{A}$  are diagonal from the very beginning.

The radial form of the measure allows one to determine the dependence of the eigenvalue density  $\rho_f(\lambda)$  on the radial profile  $f(x^2)$ . Intuitively, the reason for that stems from the fact that one can do the integration for the radial ensembles (7) in two steps: the first step is a sort of angular integration which is done for fixed  $x$  and thus is independent of the radial profile  $f(x^2)$ , and the second one is an integration over  $x$ . A short inspection of the formula (7) tells us that fixed  $x$  corresponds to fixed trace:  $\text{Tr } \mathbf{X}^\tau \mathbf{C}^{-1} \mathbf{X} \mathbf{A}^{-1}$ , and thus that we should first perform the integration over the fixed trace ensemble. We shall follow this intuition below.

Let us define a matrix  $\mathbf{x} = \mathbf{C}^{-1/2} \mathbf{X} \mathbf{A}^{-1/2}$ . Since we assumed that  $\mathbf{A}$  and  $\mathbf{C}$  are diagonal,  $\mathbf{A}^{1/2}$  and  $\mathbf{C}^{1/2}$  are also diagonal with elements being square roots of those for  $\mathbf{A}$  and  $\mathbf{C}$ . The elements of  $\mathbf{x}$  are

$$x_{it} \equiv \frac{X_{it}}{C_i A_t}. \quad (13)$$

They can be viewed as components  $x_j$ ,  $j=1, \dots, d$  of a  $d$ -dimensional Euclidean vector, where the index  $j$  is constructed from  $i$  and  $t$ . The length of this vector is

$$x^2 \equiv \sum_{j=1}^d x_j^2 = \sum_{i=1}^N \sum_{t=1}^T x_{it}^2 = \text{Tr } \mathbf{x}^\tau \mathbf{x} = \text{Tr } \mathbf{X}^\tau \mathbf{C}^{-1} \mathbf{X} \mathbf{A}^{-1}, \quad (14)$$

and thus the fixed trace matrices  $\mathbf{X}$  are mapped onto a  $d$ -dimensional sphere of the given radius  $x$ . It is convenient to parametrize the  $d$ -dimensional vector  $\mathbf{x}$  using spherical coordinates  $\mathbf{x} = x\boldsymbol{\omega}$ , where  $\boldsymbol{\omega}^2 \equiv \text{Tr } \boldsymbol{\omega}^\tau \boldsymbol{\omega} = 1$ . We can also use these coordinates to represent the matrix  $\mathbf{X}$ ,

$$\begin{aligned} \mathbf{X} &= \mathbf{C}^{1/2} \mathbf{x} \mathbf{A}^{1/2} = x \mathbf{C}^{1/2} \boldsymbol{\omega} \mathbf{A}^{1/2} = x \boldsymbol{\Omega}(\boldsymbol{\omega}), \\ \boldsymbol{\Omega}(\boldsymbol{\omega}) &\equiv \mathbf{C}^{1/2} \boldsymbol{\omega} \mathbf{A}^{1/2}, \end{aligned} \quad (15)$$

where the definition of the matrix  $\boldsymbol{\Omega}(\boldsymbol{\omega})$  is equivalent to  $\Omega_{it} \equiv C_i A_t \omega_{it}$ . While  $\boldsymbol{\omega}$  gives a point on a unit sphere in  $d$ -dimensional space,  $\boldsymbol{\Omega}(\boldsymbol{\omega})$  gives a radial projection of this point on a  $d$ -dimensional ellipsoid of fixed trace,

$$\text{Tr } \boldsymbol{\Omega}^\tau \mathbf{C}^{-1} \boldsymbol{\Omega} \mathbf{A}^{-1} = 1. \quad (16)$$

### III. ANGULAR INTEGRATION

We are now prepared to do the integration over the angular variables  $D\boldsymbol{\omega}$ . In the spherical coordinates (15) the radial measure (7) assumes a very simple form,

$$P_f(\mathbf{X}) D\mathbf{X} = \pi^{-d/2} f(x^2) x^{d-1} dx D\boldsymbol{\omega}. \quad (17)$$

The normalization factor  $\mathcal{N}^{-1}$  from Eq. (7) cancels out. The spherical coordinates  $\mathbf{X} = x\boldsymbol{\Omega}(\boldsymbol{\omega})$  allow us to write the formula for  $\rho_f(\lambda)$  in the form

$$\begin{aligned} \rho_f(\lambda) &= \pi^{-d/2} \int \rho(\mathbf{X}, \lambda) P_f(\mathbf{X}) D\mathbf{X} \\ &= \pi^{-d/2} \int D\boldsymbol{\omega} \int_0^\infty \rho(x\boldsymbol{\Omega}(\boldsymbol{\omega}), \lambda) f(x^2) x^{d-1} dx. \end{aligned} \quad (18)$$

Although the integration over the angular and the radial part cannot be entirely separated, we can partially decouple  $x$  from  $\boldsymbol{\Omega}$  in the first argument of  $\rho(x\boldsymbol{\Omega}(\boldsymbol{\omega}), \lambda)$ . It follows from (4) that the rescaling  $\mathbf{X} \rightarrow \alpha\mathbf{X}$  by a constant gives the relation

$$\rho(\alpha\mathbf{X}, \lambda) = \alpha^{-2} \rho(\mathbf{X}, \alpha^{-2}\lambda). \quad (19)$$

This observation can be used to rewrite the equation (18) in a more convenient form

$$\begin{aligned} \rho_f(\lambda) &= \pi^{-d/2} \int D\boldsymbol{\omega} \int_0^\infty \rho\left(\boldsymbol{\Omega}(\boldsymbol{\omega}), \frac{\lambda}{x^2}\right) f(x^2) x^{d-3} dx \\ &= \frac{2}{\Gamma(d/2)} \int_0^\infty \rho_*\left(\frac{\lambda}{x^2}\right) f(x^2) x^{d-3} dx, \end{aligned} \quad (20)$$

where  $\Gamma(z)$  is the Euler gamma function and

$$\rho_*(\lambda) \equiv \frac{1}{S_d} \int \rho(\boldsymbol{\Omega}(\boldsymbol{\omega}), \lambda) D\boldsymbol{\omega}. \quad (21)$$

Here  $S_d$  denotes the hypersurface area of  $d$ -dimensional sphere of radius one,  $S_d = 2\pi^{d/2}/\Gamma(\frac{d}{2})$ . As we shall see below the last expression is an eigenvalue distribution of the empirical covariance matrix for the fixed trace ensemble defined as an ensemble of matrices  $\mathbf{X}$  such that  $\text{Tr } \mathbf{X}^\tau \mathbf{C}^{-1} \mathbf{X} \mathbf{A}^{-1} = 1$ . From the structure of the equation (20) it is clear that if  $\rho_*(\lambda)$  is known then  $\rho_f(\lambda)$  can be easily calculated for any radial profile just by doing one-dimensional integral. So the question which we face now is how to determine  $\rho_*(\lambda)$  for arbitrary  $\mathbf{C}$  and  $\mathbf{A}$ . We will do this by a trick. Instead of calculating  $\rho_*(\lambda)$  directly from Eq. (21), we will express  $\rho_*(\lambda)$  by the corresponding eigenvalue density  $\rho_G(\lambda)$  for a Gaussian ensemble, whose form is known analytically [17,28]. Let us follow this strategy in the next section.

### IV. FIXED TRACE ENSEMBLE AND GAUSSIAN ENSEMBLE

The probability measure for the fixed trace ensemble is defined as

$$P_*(\mathbf{X})D\mathbf{X} = \frac{\Gamma\left(\frac{d}{2}\right)}{\mathcal{N}} \delta(\text{Tr}(\mathbf{X}^T \mathbf{C}^{-1} \mathbf{X} \mathbf{A}^{-1}) - 1) D\mathbf{X}. \quad (22)$$

In the spherical coordinates  $\boldsymbol{\omega}$  the formula reads

$$P_*(\mathbf{X})D\mathbf{X} = \frac{2}{S_d} \delta(x^2 - 1) x^{d-1} dx D\boldsymbol{\omega}.$$

One can easily check that the integration  $\rho_*(\lambda) = \int \rho(\mathbf{X}, \lambda) P_*(\mathbf{X}) D\mathbf{X}$  indeed gives (21). It is also worth noticing that the normalization condition for  $P_*(\mathbf{X})$  is fulfilled. Consider now a Gaussian ensemble,

$$P_G(\mathbf{X})D\mathbf{X} \equiv \mathcal{N}^{-1} f_G(\text{Tr} \mathbf{X}^T \mathbf{C}^{-1} \mathbf{X} \mathbf{A}^{-1}) D\mathbf{X}, \quad (23)$$

where

$$f_G(x^2) = \frac{1}{2^{d/2}} e^{-(1/2)x^2}, \quad (24)$$

for which the spectrum  $\rho_G(\lambda)$  is known or more precisely it can be easily computed numerically in the thermodynamical limit  $N, T \rightarrow \infty$  [17,34,35]. On the other hand, as we learned in the preceding section, the density of eigenvalues of the empirical covariance matrix  $\mathbf{c}$  can be found applying Eq. (20) to the Gaussian radial profile (24),

$$\rho_G(\Lambda) = \frac{2^{1-d/2}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty \rho_*\left(\frac{\Lambda}{x^2}\right) x^{d-3} e^{-(1/2)x^2} dx. \quad (25)$$

Changing the integration variable to  $y: x^2 = dy^2$  and rescaling the spectrum  $\rho_G$  by  $d: \lambda = \frac{\Lambda}{d}$  we eventually obtain

$$d\rho_G(d\lambda) = \int_0^\infty \rho_*\left(\frac{\lambda}{y^2}\right) \frac{1}{y^2} \left( \frac{2^{1-d/2} d^{d/2}}{\Gamma\left(\frac{d}{2}\right)} y^{d-1} e^{-(1/2)dy^2} \right) dy. \quad (26)$$

One can easily check that the formula in the square brackets tends to the Dirac delta for large matrices because then  $d$  goes to infinity,

$$\lim_{d \rightarrow \infty} \frac{2^{1-d/2} d^{d/2}}{\Gamma\left(\frac{d}{2}\right)} y^{d-1} e^{-(1/2)dy^2} = \delta(y - 1),$$

and thus the integrand in Eq. (26) gets localized around the value  $y=1$ . Therefore for large  $d$  we can make the following substitution:

$$\rho_*(\lambda) = d\rho_G(d\lambda). \quad (27)$$

Inserting it into Eq. (20) and changing the integration variable to  $y = \frac{d\lambda}{x^2}$  we finally obtain a central equation of this paper,

$$\rho_f(\lambda) = \frac{d^{d/2}}{\Gamma(d/2)} \lambda^{d/2-1} \int_0^\infty \rho_G(y) f\left(\frac{d\lambda}{y}\right) y^{-d/2} dy. \quad (28)$$

The meaning of this formula is the following: for any random matrix ensemble with a radial measure (7) the eigen-

value density function  $\rho_f(\lambda)$  is given by a one-dimensional integral of a combination of the corresponding Gaussian spectrum  $\rho_G(\lambda)$  and the radial profile  $f(x)$ . The equation holds in the thermodynamic limit:  $d=NT \rightarrow \infty$  and  $r=N/T = \text{const}$ . Since in this limit we are able to calculate the spectrum  $\rho_G(\lambda)$  for arbitrarily chosen  $\mathbf{A}, \mathbf{C}$ , the formula (28) gives us a powerful tool for computing spectra of various distributions. In the next section we shall apply it to the multivariate Student ensemble (9).

## V. MULTIVARIATE STUDENT ENSEMBLE

The radial profile for the Student ensemble (9) is

$$f(x^2) \equiv f_\nu(x^2) = \frac{\Gamma\left(\frac{\nu+d}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right) \nu^{d/2}} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+d)/2}. \quad (29)$$

We have chosen here the standard convention  $\sigma^2 = \nu$  since we would like to calculate the spectrum  $\rho_\nu(\lambda)$  also for  $\nu \leq 2$  (see the discussion at the end of the first section). Inserting (29) into the equation (28),

$$\begin{aligned} \rho_\nu(\lambda) &= \left(\frac{d}{\nu}\right)^{d/2} \frac{\Gamma\left(\frac{\nu+d}{2}\right)}{\Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{\nu}{2}\right)} \lambda^{d/2-1} \\ &\quad \times \int_0^\infty \rho_G(y) \left(1 + \frac{d\lambda}{\nu y}\right)^{-(\nu+d)/2} y^{-d/2} dy, \end{aligned}$$

and taking the limit  $d \rightarrow \infty$ ,

$$\begin{aligned} \lim_{d \rightarrow \infty} \left(\frac{d}{\nu}\right)^{d/2} \frac{\Gamma\left(\frac{\nu+d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} y^{-d/2} \lambda^{(d/2)-1} \left(1 + \frac{d\lambda}{\nu y}\right)^{-(\nu+d)/2} \\ = \left(\frac{\nu}{2}\right)^{\nu/2} e^{-\nu y/2\lambda} y^{\nu/2} \lambda^{-(\nu+2)/2}, \end{aligned}$$

we see that the expression for  $\rho_\nu(\lambda)$  simplifies to an expression which is independent of  $d$ ,

$$\rho_\nu(\lambda) = \frac{1}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\nu}{2}\right)^{\nu/2} \lambda^{-(\nu/2)-1} \int_0^\infty \rho_G(y) e^{-\nu y/2\lambda} y^{\nu/2} dy. \quad (30)$$

The formula (30) works for all  $\nu > 0$ . From the last equation we can infer the behavior of  $\rho_\nu(\lambda)$  for large  $\lambda$ . The function  $\rho_G(y)$  has a compact support [17,20,34], therefore for large  $\lambda$  the exponential can be approximated well by 1. The function  $\rho_\nu(\lambda)$  has thus a long tail,



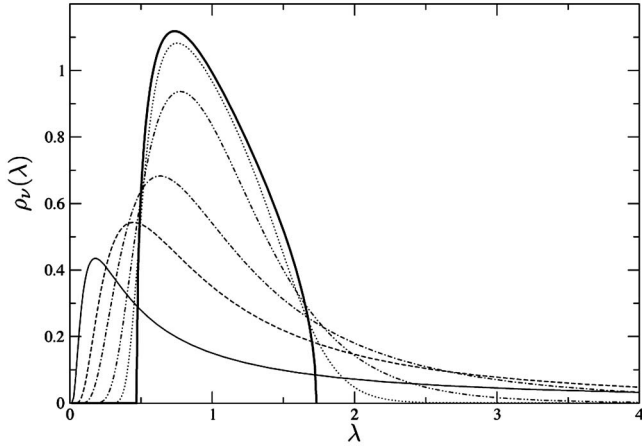


FIG. 1. Spectra of the covariance matrix  $\mathbf{c}$  for the Student distribution (9) with  $\mathbf{C}=\mathbb{1}_N$  and  $\mathbf{A}=\mathbb{1}_T$ ,  $r=N/T=0.1$ , for  $\nu=1/2, 2, 5, 20$ , and  $100$  (thin lines from solid to dotted), calculated using the formula (34) and compared to the uncorrelated Wishart (thick line). One sees that for  $\nu \rightarrow \infty$  the spectra tend to the Wishart distribution.

$$\rho_\nu(\lambda) \approx \lambda^{-(\nu/2)-1} \cdot \frac{1}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\nu}{2}\right)^{\nu/2} \int_0^\infty \rho_G(y) y^{\nu/2} dy, \quad (31)$$

where the integral does not depend on  $\lambda$ . The exponent  $-\nu/2-1$  in the above power law depends on the index  $\nu$  of the original Student distribution. The change from the power  $\nu$  to the power  $\nu/2$  comes about because  $\mathbf{c}$  is a quadratic combination of  $\mathbf{X}$ .

The power-law tail in the eigenvalue distribution (31) does not disappear in the limit of large matrices contrary to the power-law tails in the eigenvalue distribution for an ensemble of matrices whose elements are independently distributed random numbers. For such matrices, for  $\nu > 2$ , the density  $\rho(\lambda)$  falls into the Gaussian universality class and yields the Wishart spectrum [36]. One should remember that the multivariate Student distribution (9) discussed here does not describe independent degrees of freedom even for  $\mathbf{A}=\mathbb{1}_T$  and  $\mathbf{C}=\mathbb{1}_N$ , in which case the degrees of freedom are “uncorrelated” but not independent.

We have learned that the spectrum is unbounded from above. Let us now examine the lower limit of the spectrum. Rewriting Eq. (30) in the form

$$\rho_\nu(\lambda) = \frac{2\nu^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty \rho_G(2x\lambda) e^{-\nu x} x^{\nu/2} dx, \quad (32)$$

we see that as long as  $\lambda > 0$  the function  $\rho_\nu(\lambda)$  is positive since  $\rho_G(x)$  is positive on a finite support. Thus the function  $\rho_\nu(\lambda)$  vanishes only at  $\lambda=0$  and it is positive for any  $\lambda > 0$ . Contrary to the classical Wishart distribution for the Gaussian measure, the spectrum (30) spreads over the whole real positive semi-axis. On the other hand, taking the limit  $\nu \rightarrow \infty$  of Eq. (32) and using the formula

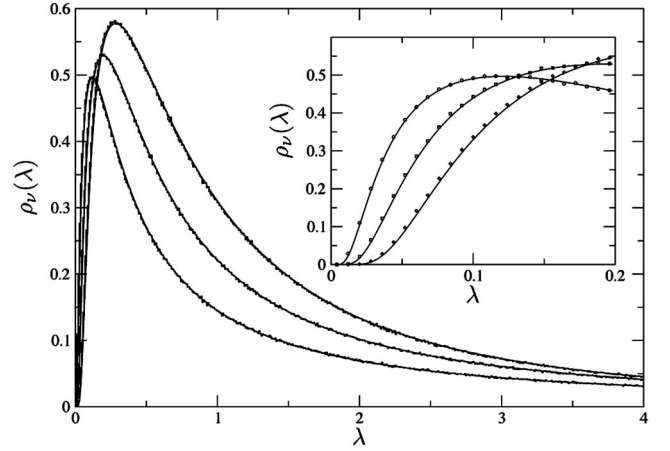


FIG. 2. Spectra of the empirical covariance matrix  $\mathbf{c}$  calculated from Eq. (34) with  $r=1/3$ , compared to experimental data (stair lines) obtained by the Monte Carlo generation of finite matrices  $N=50$ ,  $T=150$ . Inset, the left-hand part of the same distributions, points represent experimental data.

$$\lim_{\nu \rightarrow \infty} \frac{2\nu^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)} x^{\nu/2} e^{-\nu x} = \delta(x-1/2), \quad (33)$$

we obtain  $\rho_{\nu \rightarrow \infty}(\lambda) = \rho_G(\lambda)$  as expected, because in this limit the radial profile  $f_\nu(x^2)$  given by Eq. (29) for the Student distribution reduces to the Gaussian one (24).

## VI. EXAMPLES

Let us first consider the case without correlations:  $\mathbf{C}=\mathbb{1}_N$  and  $\mathbf{A}=\mathbb{1}_T$ . The spectrum of the empirical covariance for the Gaussian ensemble is given by the Wishart distribution,

$$\rho_G(\lambda) = \frac{1}{2\pi r \lambda} \sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)},$$

where  $\lambda_\pm = (1 \pm \sqrt{r})^2$  [14–16]. The corresponding spectrum (30) for the Student ensemble is then

$$\rho_\nu(\lambda) = \frac{1}{2\pi r \Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\nu}{2}\right)^{\nu/2} \lambda^{-\nu/2-1} \times \int_{\lambda_-}^{\lambda_+} \sqrt{(\lambda_+ - y)(y - \lambda_-)} e^{-\nu y/2} y^{(\nu/2)-1} dy. \quad (34)$$

The integral over  $dy$  can be easily computed numerically. Results of this computation for different values of  $\nu$  are shown in Fig. 1. For increasing  $\nu$  the spectrum  $\rho_\nu(\lambda)$  tends to the Wishart distribution but even for very large  $\nu$  it has a tail which touches  $\lambda=0$  as follows from Eq. (32). In Fig. 2 we have plotted  $\rho_\nu(\lambda)$  for  $\nu=0.5, 1$ , and  $2$  and compared them to experimental results obtained by the Monte Carlo generation of random matrices drawn from the corresponding ensemble with the probability measure (9) for which eigenvalue densities were computed by numerical diagonalization. The agreement is perfect. Actually it is even better than for the Gaussian case for the same size  $N$ .

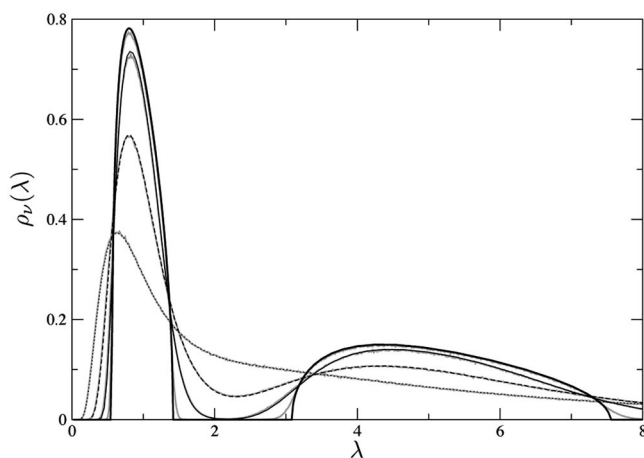


FIG. 3. Spectra  $\rho_\nu(\lambda)$  for  $\mathbf{C}$  having two distinct eigenvalues, 1 and  $\mu$  in proportion  $(1-p):p$ , calculated from Eq. (30) with  $\rho_G$  given by formula (35), with  $r=1/10$ ,  $p=1/2$ , and  $\mu=5$ . Thick solid line corresponds to the Gaussian case  $\nu \rightarrow \infty$  while thin lines correspond to  $\nu=5, 20, 100$ . These lines are compared to Monte Carlo results obtained by the generation and diagonalization of finite matrices with  $N=40$ ,  $T=400$  (gray lines), which lie almost exactly on top of them and can be hardly seen by an unarmed eye.

As a second example we consider the case when  $\mathbf{C}$  has two distinct eigenvalues  $\lambda_1$  and  $\lambda_2$  with degeneracies:  $(1-p)N$  for  $\lambda_1$  and  $pN$  for  $\lambda_2$ , where  $0 \leq p \leq 1$ . Such a covariance matrix can be used to model the simplest effect of sectorization on a stock exchange. For example, if all diagonal elements of the matrix  $\mathbf{C}$  are equal 1 and all off-diagonal are equal  $\rho_0$  ( $0 < \rho_0 < 1$ ) the model can be used to mimic a collective behavior on the market [9,10]. In this case  $\lambda_1=1-\rho_0$  has a degeneracy  $N-1$  and  $\lambda_2=1+(N-1)\rho_0$  is nondegenerated, hence  $p=1/N$ . The eigenvector corresponding to the larger eigenvalue  $\lambda_2$  can be thought of as describing the correlations of all stocks. For our purposes it is however more convenient to set  $\lambda_1=1$  and  $\lambda_2 \equiv \mu$  and  $p$  being an arbitrary number between zero and one. The corresponding Wishart spectrum  $\rho_G(\lambda)$  can be obtained by solving equations given by a conformal map [20]. The resulting spectrum has the form

$$\rho_G(\lambda) = \frac{1}{\pi} \left| \operatorname{Im} \frac{M[Z(\lambda)]}{\lambda} \right|, \quad (35)$$

where

$$M(Z) = \frac{1-p}{Z-1} + \frac{p\mu}{Z-\mu}, \quad (36)$$

$$Z(\lambda) = -\frac{a}{3} + \frac{(1-i\sqrt{3})(3b-a^2)}{3 \times 2^{2/3}E} - \frac{(1+i\sqrt{3})E}{6 \times 2^{1/3}}, \quad (37)$$

$$E = (3\sqrt{3}\sqrt{27c^2 - 18abc + 4a^3c + 4b^3 - a^2b^2} - 27c + 9ab - 2a^3)^{1/3}, \quad (38)$$

where  $a=r-1-pr-\mu(1-pr)-\lambda$ ,  $b=\lambda(\mu+1)-\mu(1-r)$  and  $c=-\lambda\mu$ . Inserting the above formula into Eq. (30) we obtain an integral, which can be computed numerically for arbitrary  $r, \mu, p$ . In Fig. 3 we show examples of this computation for different values of the index  $\nu$ . In the same figure we compare the analytic results with those obtained by the Monte Carlo generation and numerical diagonalization of random matrices for  $N=40$ ,  $T=400$ . As before, the agreement between the analytic and Monte Carlo results is perfect. We see that the effect on the spectrum of introducing heavy tails increases with decreasing  $\nu$ . When  $\nu$  is decreasing from infinity to zero the two disjoint islands of the distribution develop a bridge to eventually end up as a distribution having only one connected component.

## VII. SUMMARY

In the paper we have developed a method for computing spectral densities of empirical covariance matrices for a wide class of “quasi-Wishart” ensembles with radial probability measures. In particular we have applied this method to determine the spectral density of the empirical covariance matrix for heavy tailed data described by a Student multivariate distribution. We have shown that the spectrum  $\rho(\lambda)$  decays like  $\lambda^{-\nu/2-1}$  where  $\nu$  is the index of Student distribution. The case of  $\nu=3$  is of particular importance since it can be used in modeling stock markets. The eigenvalue density spreads over the whole positive semiaxis in contrast to the Wishart spectrum which has a finite support.

We have also derived a general formula for the eigenvalue spectrum of the empirical covariance matrix for radial ensembles. The spectrum is given by a one-dimensional integral, which can be easily computed numerically. The method works also in the case of correlated assets.

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